UDC 539.3

ON THE SOLUTION OF PROBLEMS OF THE THEORY OF ELASTICITY BY THE METHOD OF FACTORIZATION

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G. Ia. POPOV
(Odessa)

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For the solution of problems which are solvable by the method of factorization [1], different approximate methods have been suggested in recent years: approximate factorization [1], variation and projection methods [2], and finally, the method of orthogonal polynomials [3]. The aim of this paper is to indicate (remaining within the limits of the exact solution) a simple method of transforming the formulas which give the solution of the factorization problem to a form which is more convenient for computations and on the basis of this method to indicate the exact solutions of the Wiener-Hopf-Fock integral equations of the second and the first kind in a form which is more convenient for carrying out the calculations. The computational efficiency of the suggested method is illustrated in [4].

1. According to the terminology of [5], by factorization, i.e. the representation of a function $G(x)$, given on the closed line $(-\infty, \infty)$ and not vanishing there, in the form

$$G(x) = G_+(x)\, G_-(x) \qquad (-\infty < x < \infty) \tag{1.1}$$

where $G_+(x)$ and $G_-(x)$ are regular functions, different from zero in the upper and lower half-planes and continuous, including the boundary. Without loss of generality, we will assume that

$$G(\pm\infty) = 1, \qquad G_\pm(\infty) = 1 \tag{1.2}$$

In addition, we will assume that the logarithmic derivative of the function $G(x)$ is integrable on the real axis.

The solution of the factorization problem is given by the formula [5]

$$\ln G_\pm(z) = \pm\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln G(x)\,dx}{x-z} \qquad \begin{pmatrix}\operatorname{Im} z > 0\\ \operatorname{Im} z < 0\end{pmatrix} \tag{1.3}$$

We consider the transformation of this formula. All the operations will be performed only for the formula for $G_+(z)$, since for $G_-(z)$ they are completely similar; moreover, for the most frequent case of an even function $G(x)$, we have [5] the relation $G_-(z) = G_+(-z)$.

Mapping the upper half-plane of the variable z from the formula (1.3) onto the unit circle of the variable ζ and the real axis $(-\infty < x < \infty)$ onto the unit circumference γ, instead of (1.3) we have

$$\ln G_+\left(-i\frac{\zeta+1}{\zeta-1}\right) = h(\zeta) = \frac{1}{2\pi i}\int_\gamma \ln G\left(-i\frac{\sigma+1}{\sigma-1}\right)\frac{d\sigma}{\sigma-\zeta} - \frac{1}{2\pi i}\int_\gamma \ln G\left(-i\frac{\sigma+1}{\sigma-1}\right)\frac{d\sigma}{\sigma-1} \tag{1.4}$$

Applying to the first integral the method described in [6], we obtain

$$h(\zeta) = \sum_{n=0}^{\infty} h_n \zeta^n \qquad (1.5)$$

$$h_0 = -\frac{1}{4\pi i}\int_0^{2\pi} \frac{\ln G(-\mathrm{ctg}\,{}^1/_2\varphi)\,d\varphi}{e^{1/_2 i\varphi}\sin {}^1/_2\varphi} = \frac{1}{2\pi i}\int_0^{\pi} \frac{\ln G(-\mathrm{ctg}\,\varphi)\,d\varphi}{e^{-i\varphi}\sin\varphi}$$

$$h_n = \frac{1}{2\pi}\int_0^{2\pi} \frac{\ln G(-\mathrm{ctg}\,{}^1/_2\varphi)\,d\varphi}{e^{in\varphi}} = \frac{1}{2\pi i n}\int_0^{2\pi} \frac{d\,[\ln G(-\mathrm{ctg}\,\varphi)]}{e^{in\varphi}}$$

Thus, the coefficients of the Maclaurin series, the functions represent, except for constants, the Fourier coefficients of the specified function and for their computation we can, in general, make use of the known formulas of trigonometric interpolation [7].

In special cases the formula [4]

$$h_n = \frac{1}{2\pi i n}\int_{-\infty}^{\infty} \left(\frac{x+i}{x-i}\right)^n d\,[\ln G(x)] \qquad (1.6)$$

may be useful, which is obtained from the second formula for h_n from (1.5) by conversion to integration along the unit circumference and then the mapping of the latter into the real axis. In the important special case when $G(x)$ is an even function, the formulas (1.5) become simpler

$$h_0 = \frac{1}{2\pi}\int_0^{\pi} \ln G\left(\mathrm{tg}\,\frac{1}{2}\,\theta\right) d\theta, \qquad h_n = \frac{(-1)^{n+1}}{n\pi}\int_0^{\pi} \sin n\theta\, d\left[\ln G\left(\mathrm{tg}\,\frac{1}{2}\,\theta\right)\right] \qquad (1.7)$$

In order to obtain these formulas it is necessary to perform the substitution $\varphi = \pi + \theta$ in the first integrals which define h_0 and h_n in (1.5) and to make use of the fact that $G(x)$ is even.

Thus, the desired function $G_+(z)$ which occurs in the factorization of the function $G(x)$, will be determined by the formula

$$G_+\left(-i\,\frac{\zeta+1}{\zeta-1}\right) = \exp\left(\sum_{h=0}^{\infty} h_n \zeta^n\right) = g(\zeta) \qquad (|\zeta| \leqslant 1) \qquad (1.8)$$

Such a representation is not entirely convenient if the function has to be integrated along the real axis. In this case it is of interest to obtain formulas for the coefficients of the Maclaurin series of the function

$$g(\zeta) = \sum_{n=0}^{\infty} g_n \zeta^n \qquad (1.9)$$

Obviously, $g_0 = e^{h_0}$.

For the computation of the remaining coefficients we proceed in the following manner. On the basis of (1.8) we have $h(\zeta) = \ln g(\zeta)$ and therefore $g'(\zeta) = h'(\zeta)\,g(\zeta)$. Substituting here the expressions

$$g'(\zeta) = \sum_{m=0}^{\infty} (m+1)\, g_{m+1}\zeta^m, \qquad h'(\zeta) = \sum_{m=0}^{\infty} (m+1)\, h_{m+1}\zeta^m$$

and also the expansions (1.9) and equating the coefficients, we obtain the following recursion formula:

$$g_{n+1}=\sum_{m=0}^{n}\frac{n-m+1}{n+1}h_{n-m+1}g_m \qquad (n=0,1,2\ldots) \tag{1.10}$$

Below we give an illustration and the development of the suggested method as applied in connection with the Wiener-Hopf-Fock integral equation of the second and the first kind.

2. First we consider the equation of the second kind with a particular right-hand side

$$\chi_p(x)-\int_0^\infty k(x-y)\chi_p(y)\,dy=e^{ipx} \qquad (\operatorname{Im} p>0;\ K(x)=K(-x)) \tag{2.1}$$

assuming, for the sake of simplicity, that the kernel function is even. The solution of Eq. (2.1) has the form [5]

$$\chi_p(x)=-\frac{G_+(p)}{2\pi i}\int_{-\infty}^{\infty}\frac{G_+(s)\,e^{-isx}\,ds}{s+p} \tag{2.2}$$

In the case under consideration the function $G_+(z)$ factorizes the function

$$G(x)=[1-K(x)]^{-1}=G_+(x)\,G_-(x),\quad K(x)=\int_{-\infty}^{\infty}k(s)\,e^{isx}\,ds \tag{2.3}$$

On the basis of (1.8), (1.9) we have the expansion

$$G_+(s)=\sum_{n=0}^{\infty}g_n\left(\frac{s-i}{s+i}\right)^n \tag{2.4}$$

which inserted into (2.2) gives

$$\chi_p(x)=[1-K(p)]^{-1}e^{ipx}+G_+(p)\sum_{n=0}^{\infty}g_n\operatorname{Res}\left[\left(\frac{s-i}{s+i}\right)^n\frac{e^{-isx}}{s+p}\right]$$

There is another way to proceed. We represent the function (2.2) in a series of Laguerre polynomials

$$\chi_p(x)=2e^{-x}\sum_{j=0}^{\infty}\chi_j(p)\,L_j(2x) \tag{2.5}$$

then

$$\chi_j(p)=\int_0^\infty e^{-x}L_j(2x)\,\chi_p(x)\,dx=\frac{G_+(p)}{2\pi}\int_{-\infty}^{\infty}\frac{G_+(s)\,(s+i)^j\,ds}{(s+p)(s-i)^{j+1}} \tag{2.6}$$

For the computation of the last integral we map the upper half-plane of the variable s onto the unit circle of the variable t, i.e. we perform the substitution $s=-i(t+1)(t-1)^{-1}$. The integral along the unit circle obtained in this way is computed with the aid of the residue theorem and the expansions (1.8), (1.9). As a result we obtain the formula

$$\chi_j(p)=i\,\frac{G_+(p)}{p+i}\sum_{r=0}^{j}g_r\left(\frac{p-i}{p+i}\right)^{j-r} \tag{2.7}$$

Assume now that the integral equation (2.1) has the form

$$\chi(x)-\int_0^\infty k(x-y)\,\chi(y)\,dy=f(x) \qquad (x\geqslant 0) \tag{2.8}$$

Then, if

$$F(u)=\int_0^\infty e^{ixu}f(x)\,dx \tag{2.9}$$

the solution has the form

$$\chi(x) = \frac{1}{4\pi} \int_{-\infty}^{\infty} F(-p)\chi_p(x)\,dp \tag{2.10}$$

or on the basis of (2.5) and (2.7)

$$\chi(x) = 2e^{-x} \sum_{j=0}^{\infty} F_j L_j(2x), \qquad F_j = -\sum_{r=0}^{j} g_r I_{j-r} \tag{2.11}$$

$$I_m = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(-p)\,G+(p)}{p+i} \left(\frac{p-i}{p+1}\right)^m dp \tag{2.12}$$

For some particular cases of right-hand sides in (2.8), this integral can be easily evaluated by the residue theorem. In general we proceed in the following manner. Making use of the expansion (2.4), we write

$$I_k = \sum_{n=0}^{\infty} g_n f_{n+k}, \qquad f_m = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(-p)}{p+i} \left(\frac{p-i}{p+i}\right)^m dp \tag{2.13}$$

The substitution $p = \operatorname{ctg} {}^1\!/_2\, \varphi$ leads to the formula

$$f_m = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\varphi} F^* \left(-\operatorname{ctg} \frac{1}{2}\varphi\right) d\varphi \tag{2.14}$$

$$2iF^*(x) = (x - i)\,F(-x)$$

i.e. the coefficients f_m represent exactly the Fourier coefficients.

3. In the problems of the theory of elasticity we frequently encounter integral equations of the first kind

$$\int_0^{\infty} k(x-y)\,\chi(y)\,dy = f(x) \qquad (x \geqslant 0,\ k(x) = k(-x)) \tag{3.1}$$

We will assume that the Fourier transform of the kernel function is differentiable on the real axis, it is different from zero there and at infinity it has the asymptotics

$$K(u) = \gamma u^{2\mu+1}[1 + O(u^{-1})] \qquad (u \to \infty,\ |\mu| < {}^1\!/_2) \tag{3.2}$$

Without loss of generality, we will assume everywhere in the sequel that $\gamma = 1$. For the integral equation with the particular right-hand side

$$\int_0^{\infty} k(x-y)\,\chi_p(y)\,dy = e^{ipx} \qquad (x,\ \operatorname{Im} p \geqslant 0) \tag{3.3}$$

we have, as before, the formula [5, 8]

$$\chi_p(x) = -\frac{K_+(p)}{2\pi i} \int_{-\infty}^{\infty} \frac{K_+(s)\,e^{-isx}\,ds}{s+p} \tag{3.4}$$

where $K_+(z)$ is a regular function (except at the point ∞), different from zero in the upper half-plane and satisfying the functional equation

$$K^{-1}(z) = K_+(z)\,K_-(z) \qquad (-\infty < z < \infty) \tag{3.5}$$

Taking into account the asymptotics (3.2), we represent $K(z)$ in the form

$$K^{-1}(z) = (z^2+1)^{1/2-\mu}\,G(z), \qquad G^{-1}(z) = (z^2+1)^{1/2-\mu}\,K(z) \tag{3.6}$$

The function $G(z)$ can be factored out, making use of the formula (1.3), and by the same token the solution of equation (3.5) is obtained in the form

$$K_+(z) = (1 - iz)^{1/2-\mu} G_+(z) \tag{3.7}$$

In this case for $G_+(z)$ the formula (2.4) should be used.

For some particular cases of kernel functions, the integral in (3.4) can be reduced by the methods of contour integration [8] to a form which is convenient for computations. In the general case, the following method is useful. Taking into account the character of the singularity of the solution of the equation (3.3) at zero, which follows [3] from the asymptotics (3.2), we represent it in the form of the following series of Chebyshev-Laguerre polynomials:

$$\chi_p(x) = \sum_{m=0}^{\infty} \frac{2e^{-x}\chi_m(p) L_m^{\mu-1/2}(2x)}{x^{1/2-\mu}\mu_m}, \qquad \mu_m = \frac{\Gamma(1/2+\mu+m)}{m!} \tag{3.8}$$

In this case

$$\chi_m(p) = -\frac{2^{\mu-1/2} K_+(p)}{2\pi i} \int_{-\infty}^{\infty} \frac{K_+(s) I_m^-(s)\, ds}{s+p} \tag{3.9}$$

$$I_m^-(s) = \int_0^{\infty} \frac{L_m^{\mu-1/2}(2x)\, dx}{e^{x(1+is)}} = \frac{1}{i} \sum_{k=0}^{m} \frac{(\mu - 1/2)_{m-k} (s+i)^k}{(m-k)!\,(s-i)^{k+1}} \tag{3.10}$$

Then, in the formula (3.7) we perform the substitution $z = -i(\zeta+1)(\zeta-1)^{-1}$, and afterwards we expand the left-hand side in a Maclaurin series with respect to ζ, making use of (1.8) and (1.9). As a result we have

$$K_+\left(-i\frac{\zeta+1}{\zeta-1}\right) = \sum_{i=0}^{\infty} g_j^* \zeta^j \tag{3.11}$$

$$g_j^* = 2^{1/2-\mu} \sum_{r=0}^{j} \frac{(1/2-\mu)_r g_{j-r}}{r!}$$

We substitute now (3.10) and (3.11) into (3.9) and we compute the obtained integral by the same method as the integral in (2.6). As a result we obtain

$$\chi_m(p) = \frac{i2^{\mu-1/2} K_+(p)}{p+i} \sum_{k=0}^{m} \frac{(\mu-1/2)_{m-k}}{(m-k)!} \sum_{n=0}^{k} g_n^* \left(\frac{p-i}{p+i}\right)^{k-n}. \tag{3.12}$$

Substituting here the expression for g_n^* from (3.11) and making use of the relation

$$C_n = \sum_{j=0}^{n} \frac{(\mu-1/2)_{n-j}(1/2-\mu)_j}{(n-j)!\,j!} = \begin{cases} 1, & n=0 \\ 0, & n=1,2\ldots \end{cases} \tag{3.13}$$

we finally obtain

$$\chi_m(p) = \frac{iK_+(p)}{p+i} \sum_{j=0}^{m} \left(\frac{p-i}{p+i}\right)^j g_{m-j} \tag{3.14}$$

The validity of the relation (3.13) can be seen if we consider that C_n is the coefficient of the Maclaurin series of the function $(1-x)^{\mu-1/2}(1-x)^{1/2-\mu}$, obtained from the multiplication of the Maclaurin series for each of the factors.

For the solution of the integral equation (3.1), as in the case of the equation of the second kind, the formula (2.10) holds, which, making use of (3.8) and (3.14), can be reduced to the form

$$\chi(x) = \sum_{m=0}^{\infty} \frac{2F_m L_m^{\mu-1/2}(2x)}{e^x x^{1/2-\mu} \mu_m}, \qquad F_n = -\sum_{r=0}^{n} g_r I_{m-r} \tag{3.15}$$

For the integral I_m we have the formula (2.12) with the replacement of $G_+(p)$ by $K_+(p)$. In this case, for the latter one we can, apart of (3.7), make use of the expansion

$$K_+(p) = \sum_{m=0}^{\infty} g_m^* \left(\frac{p-i}{p+i}\right)^m \tag{3.16}$$

which follows from (3.11). Its use leads to the formulas (2.13), (2.14) and (2.16) with the replacement of g_n by g_n^*.

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OPERATOR METHOD OF INVESTIGATING THE STRAIN OF A HOLLOW SPHERE WITH DIFFERENT CREEP IN TENSION AND COMPRESSION

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M. I. ROZOVSKII
(Dnepropetrovsk)

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The polar symmetric strain of a viscoelastic thick-walled hollow sphere, whose material possesses the property of different resistivity under tension and compression is considered. The vessel is subjected to internal pressure p and external tension p which are distributed uniformly over the surfaces $r = a$ and $r = b$ $(a < b)$. Because of the above, the vessel is separated into two parts by a spherical surface of radius $r = \rho$, which is independent of the quantity p during solution of the corresponding elastic problem [1] even when p varies with time t.